

Mutation and $\mathrm{SL}(2, \mathbf{C})$ -Reidemeister torsion for hyperbolic knots.

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Abstract

Given a hyperbolic knot, we prove that the Reidemeister torsion of any lift of the holonomy to $\mathrm{SL}(2, \mathbf{C})$ is invariant under mutation along a Conway sphere.

1 Introduction

Let $K \subset S^3$ be a hyperbolic knot and $C \subset S^3$ a Conway sphere. Namely C intersects transversally K in four points. We write $\tau = \tau_i : (C, C \cap K) \rightarrow (C, C \cap K)$ to denote any of the three involutions in Figure 1, for $i = 1, 2, 3$.

The knot $K^\tau \subset S^3$ obtained by cutting along C and gluing again after composing with τ is called the mutant knot. We are interested in comparing invariants of K and K^τ , thus we may assume that, if $M = S^3 \setminus \mathcal{N}(K)$ denotes the knot exterior, then the surface $S = C \cap M$ is essential in M .

Ruberman [15] showed that K^τ is also hyperbolic, and that $M^\tau = S^3 \setminus \mathcal{N}(K^\tau)$ has the same volume as $M = S^3 \setminus \mathcal{N}(K)$. See [5, 11] for a recent account on invariants that distinguish or not K from K^τ .

Let $\rho : \pi_1(M) \rightarrow \mathrm{SL}(2, \mathbf{C})$ be a lift of the holonomy. If $\mu \in \pi_1(M)$ is a meridian, then $\mathrm{trace}(\rho(\mu)) = \pm 2$, and there are precisely two lifts of the holonomy up to conjugation, one with $\mathrm{trace}(\rho(\mu)) = +2$ and another with $\mathrm{trace}(\rho(\mu)) = -2$.

By [9], ρ is acyclic, namely the homology and cohomology of M with coefficients twisted by ρ vanish, hence the Reidemeister torsion $\mathrm{tor}(M, \rho)$ is well defined. Moreover as the dimension of \mathbf{C}^2 is even, there is no sign indeterminacy, thus $\mathrm{tor}(M, \rho)$ is a well defined nonzero complex number, independent of the conjugacy class of ρ . Hence these torsions are two topological invariants of the hyperbolic knot.

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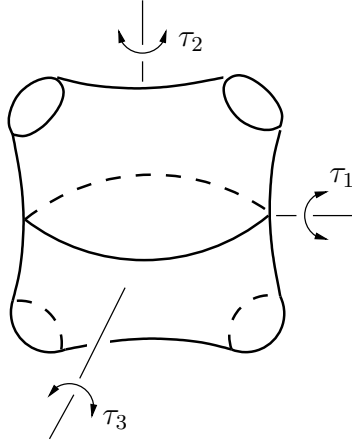


Figure 1: The involutions on the Conway sphere

Theorem 1.1. *Let K , M , τ and M^τ be as above. Let $\rho : \pi_1(M) \rightarrow \mathrm{SL}(2, \mathbf{C})$ and $\rho^\tau : \pi_1(M^\tau) \rightarrow \mathrm{SL}(2, \mathbf{C})$ be lifts of the holonomy, with $\mathrm{trace}(\rho(\mu)) = \mathrm{trace}(\rho^\tau(\mu))$. Then*

$$\mathrm{tor}(M, \rho) = \mathrm{tor}(M^\tau, \rho^\tau).$$

This is not true for any representation of $\pi_1(M)$. Wada proved in [20] that the twisted Alexander polynomials could be used to distinguish mutant knots. N. Dunfield, S. Friedl and N. Jackson [4] computed the torsion for the representation ρ twisted by the abelianization map (namely, the corresponding twisted Alexander polynomials) and proved that it distinguishes mutant knots. However, the evaluation at ± 1 of these polynomials gives numerical evidence of Theorem 1.1.

In [9] we prove that when we consider $\sigma_{2n} : \mathrm{SL}(2, \mathbf{C}) \rightarrow \mathrm{SL}(2n, \mathbf{C})$ the $2n$ -dimensional irreducible representation, then the composition $\sigma_{2n} \circ \rho$ is acyclic, thus its torsion is well defined. We have computed that the torsion of $\sigma_4 \circ \rho$ distinguishes the Conway and the Kinoshita-Terasaka mutants, see Section 6.

The paper is organized as follows. In Section 2, we discuss the basic constructions for Reidemeister torsion and representations of mutants, and we give a sufficient criterion in Proposition 2.3 for invariance of the torsion under mutation. The sufficient criterion of Proposition 2.3 is stated in terms of the action of τ on the cohomology of S with twisted coefficients. This is applied for the proof when $\mathrm{trace}(\rho(\mu)) = -2$ in Section 3. The proof when $\mathrm{trace}(\rho(\mu)) = +2$ in Section 4 is different, because the involved cohomology groups are different in each situation. In Section 5 we compute an example, the Kinoshita-Terasaka and Conway mutants, and Section 6 is devoted to further discussion.

2 Mutation

Let B_1 and B_2 denote the components of $S^3 \setminus \mathcal{N}(C)$, so that the pairs $(B_i, B_i \cap K)$ are tangles with two strings. The exterior of the knot is denoted by

$$M = S^3 \setminus \mathcal{N}(K).$$

We also denote

$$S = C \cap M, \quad M_1 = M \cap B_1, \quad \text{and} \quad M_2 = M \cap B_2.$$

Write a commutative diagram for the inclusions:

$$\begin{array}{ccc} C & \xrightarrow{i_1} & B_1 \\ i_2 \downarrow & & \downarrow \\ B_2 & \longrightarrow & S^3 \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \xrightarrow{i_1} & M_1 \\ i_2 \downarrow & & \downarrow \\ M_2 & \longrightarrow & M \end{array}$$

so that $\pi_1(M)$ is an amalgamated product

$$\pi_1(M) = \pi_1(M_1) *_{\pi_1(S)} \pi_1(M_2).$$

Let ρ_0 , ρ_1 and ρ_2 denote the restriction of ρ to $\pi_1(S)$, $\pi_1(M_1)$, and $\pi_1(M_2)$, respectively, so that

$$\rho_1 \circ i_{1*} = \rho_2 \circ i_{2*} = \rho_0.$$

Using the notation

$$\rho_0^a(\gamma) = a\rho_0(\gamma)a^{-1}, \quad \forall \gamma \in \pi_1(S),$$

there exists $a \in \text{SL}(2, \mathbf{C})$ which is unique up to sign [2, 15, 18] such that

$$\rho_0^a \circ \tau_* = \tau^* \circ \rho_0^a = \rho_0.$$

Notice that $a \in \text{SL}(2, \mathbf{C})$ corresponds to a rotation of order two in hyperbolic space, therefore, a is conjugate to

$$a \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (1)$$

To construct the representation of $\pi_1(M^\tau)$, we also use the amalgamated product structure with the same inclusion i_1 , but with $i_2 \circ \tau$ instead of i_2 . The representation $\rho^\tau : \pi_1(M^\tau) \rightarrow \text{SL}(2, \mathbf{C})$ is then defined by

$$\rho^\tau|_{\pi_1(M_1)} = \rho_1 \quad \text{and} \quad \rho^\tau|_{\pi_1(M_2)} = \rho_2^a,$$

because $\rho_2^a \circ (i_2 \circ \tau)_* = \rho_0^a \circ \tau_* = \rho_0 = \rho_1 \circ i_{1*}$.

2.1 Cohomology with twisted coefficients

To set notation, we recall the basic construction of cohomology with twisted coefficients. Let X be a compact CW -complex and $\rho : \pi_1(X) \rightarrow \mathrm{SL}(2, \mathbf{C})$ a representation. The cellular chains of its universal covering are denoted by $C_*(\tilde{X}; \mathbf{Z})$, which is a chain complex of left $\mathbf{Z}[\pi_1(X)]$ -modules of finite type. The cochains with twisted coefficients are then

$$C^*(X; \rho) = \mathrm{hom}_{\mathbf{Z}[\pi_1(X)]}(C_*(\tilde{X}; \mathbf{Z}), \mathbf{C}_\rho^2),$$

where $\mathbf{C}^2 = \mathbf{C}_\rho^2$ is viewed as a left $\mathbf{Z}[\pi_1(X)]$ -module by the action induced by ρ . The corresponding cohomology groups are denoted by

$$H^*(|X|; \rho),$$

as they only depend on the underlying topological space $|X|$ of the CW -complex X . We shall mainly work with aspherical spaces, in this case the homology or cohomology of X with twisted coefficients is naturally isomorphic to the group cohomology of $\pi_1(X)$.

We shall also be interested in the de Rham cohomology. Assuming N is a smooth manifold, let $E(\rho) = \tilde{N} \times \mathbf{C}^2 / \pi_1(N)$ denote the flat bundle with monodromy ρ . The space of p -forms valued on $E(\rho)$ is $\Omega^p(N; \rho) = \Gamma(\Lambda^p T N^* \otimes E(\rho))$. The de Rham cohomology of $(\Omega^*(N; \rho), d)$ is naturally isomorphic to $H^*(N; \rho)$,

Many properties of cohomology without coefficients hold true when we have twisted coefficients: Mayer-Vietoris, long exact sequence of the pair, etc. Poincaré duality is discussed in Section 3.1.

2.2 Mayer-Vietoris exact sequences with twisted coefficients

We will use Mayer-Vietoris for the pair (M_1, M_2) to compute the torsion of M and of M^τ . For this, we need to compute some cohomology groups. We start with the planar surface with four boundary components $S = M_1 \cap M_2$:

Lemma 2.1. $H^i(S; \rho_0) = 0$ for $i \neq 1$ and $H^1(S; \rho_0) \cong \mathbf{C}^4$.

Proof. Firstly $H^0(S, \rho_0) \cong H^0(\pi_1(S), \rho_0)$ is isomorphic to the subspace of \mathbf{C}^2 of elements that are fixed by $\rho_0(\pi_1(S))$, hence it vanishes because ρ_0 is an irreducible representation. On the other hand, $H^i(S, \rho_0) = 0$ for $i \geq 2$, because S has the homotopy type of a graph. Finally

$$\dim_{\mathbf{C}} H^1(S; \rho_0) = -\chi(S) \dim(\mathbf{C}^2) = 4.$$

□

Lemma 2.2. For $k = 1, 2$, $H^j(M_k; \rho_0) = 0$ for $j \neq 1$ and $H^1(M_k; \rho_0) = \mathbf{C}^2$.

Proof. By Mayer-Vietoris, and using the fact that $H^*(M; \rho) = 0$, then

$$H^j(M_1; \rho_1) \oplus H^j(M_2; \rho_2) \cong H^j(S; \rho_0).$$

The lemma follows from Lemma 2.1, because $\chi(M_k) = \frac{1}{2}\chi(S) = 1$. \square

Mayer-Vietoris for M and M^τ give the isomorphisms:

$$i_1^* \oplus i_2^* : H^1(M_1; \rho_1) \oplus H^1(M_2; \rho_2) \rightarrow H^1(S; \rho_0), \quad (2)$$

$$i_1^* \oplus (i_2 \circ \tau)^* : H^1(M_1; \rho_1) \oplus H^1(M_2; \rho_2^a) \rightarrow H^1(S; \rho_0). \quad (3)$$

In order to relate $H^*(M_2; \rho_2)$ and $H^*(M_2; \rho_2^a)$, we use the composition with a , where ρ_2^a denotes ρ_2 conjugated by a . Namely, recall that

$$C^*(M_2; \rho_2) = \text{hom}_{\mathbf{Z}[\pi_1(M_2)]}(C_*(\widetilde{M}_2; \mathbf{Z}), \mathbf{C}_{\rho_2}^2).$$

Define:

$$\begin{aligned} a_* : C^*(M_2; \rho_2) &\rightarrow C^*(M_2; \rho_2^a) \\ \theta &\mapsto a \circ \theta \end{aligned}$$

It is straightforward to check that this defines an isomorphism of complexes. Thus $a_* : H^1(M_2; \rho_2) \rightarrow H^1(M_2; \rho_2^a)$ is an isomorphism and we have a commutative diagram:

$$\begin{array}{ccc} H^1(M_2; \rho_2) & \xrightarrow{i_2^*} & H^1(S; \rho_2) \\ a_* \downarrow & & \downarrow a_* \circ \tau^* \\ H^1(M_2; \rho_2^a) & \xrightarrow{(i_2 \circ \tau)^*} & H^1(S; \rho_0) \end{array}$$

Write $\tau_a^* = a_* \circ \tau^* = \tau^* \circ a_*$. Since $a^2 = -\text{Id}$ (see Equation (1)) and since $\tau^2 = \text{Id}$, we have:

$$(\tau_a^*)^2 = -\text{Id}. \quad (4)$$

2.3 Reidemeister torsions

Let X be a compact CW-complex equipped with a representation

$$\rho : \pi_1(X) \rightarrow \text{SL}(2, \mathbf{C}).$$

When $H^*(|X|; \rho) = 0$, the Reidemeister torsion can be defined, and it is an invariant of X , up to subdivision, and the conjugacy class of ρ . We will not recall the definition, that can be found in [10, 19], for instance. There are two main issues for the torsion we are interested in. Firstly, the torsion is only defined up to sign, but since we consider a two dimensional vector space, it is sign defined, hence a nonzero complex number. Equivalently, any choice of homology orientation for Turaev's refined torsion [19] gives the same result. Secondly, since we are working with three and two-dimensional manifolds,

the PL-structure is not relevant. Thus, for a two and three-dimensional manifold X and an acyclic representation $\rho : \pi_1(X) \rightarrow \mathrm{SL}(2, \mathbf{C})$, its torsion is denoted by

$$\mathrm{tor}(|X|, \rho) \in \mathbf{C} \setminus \{0\}.$$

When ρ is not acyclic, then we can also use the Reidemeister torsion provided we specify a basis for $H^*(|X|; \rho)$.

Choose b_i a basis for $H^1(M_i; \rho_i)$ as \mathbf{C} -vector space. In particular $a_*(b_2)$ is a basis for $H^1(M_2; \rho_2^a)$. By Milnor's formula [10] for the torsion of a long exact sequence applied to (2) and (3):

$$\begin{aligned} \mathrm{tor}(M, \rho) &= \pm \frac{\mathrm{tor}(M_1, \rho_1, b_1) \mathrm{tor}(M_2, \rho_2, b_2)}{\mathrm{tor}(S, \rho_0, i_1^*(b_1) \sqcup i_2^*(b_2))} \\ \mathrm{tor}(M^\tau, \rho^\tau) &= \pm \frac{\mathrm{tor}(M_1, \rho_1, b_1) \mathrm{tor}(M_2, \rho_2^a, a_*(b_2))}{\mathrm{tor}(S, \rho_0, i_1^*(b_1) \sqcup (i_2^\tau)^* a^*(b_2))}. \end{aligned}$$

Here \sqcup denotes the disjoint union of basis. Notice that Milnor works with torsions up to sign in [10], but its formalism applies even with sign. Since $\tau_a^* = \tau^* \circ a_*$, we deduce

$$\frac{\mathrm{tor}(M, \rho)}{\mathrm{tor}(M^\tau, \rho^\tau)} = \det(i_1^*(b_1) \sqcup \tau_a^*(i_2^*(b_2)), i_1^*(b_1) \sqcup i_2^*(b_2)). \quad (5)$$

Namely, the determinant of the matrix whose entries are the coefficients of the basis $i_1^*(b_1) \sqcup \tau_a^*(i_2^*(b_2))$ with respect to $i_1^*(b_1) \sqcup i_2^*(b_2)$.

The following is a sufficient criterion for invariance of torsion with respect to mutation.

Proposition 2.3. *If $\tau_a^* : H^1(S; \rho_0) \rightarrow H^1(S; \rho_0)$ leaves invariant the image of $i_2^* : H^1(M_2; \rho_2) \rightarrow H^1(S, \rho_0)$, then $\mathrm{tor}(M, \rho) = \pm \mathrm{tor}(M^\tau, \rho^\tau)$.*

Proof. Since $(\tau_a^*)^2 = -\mathrm{Id}$ by Formula (4), τ_a^* diagonalizes with eigenvalues $\pm i$. Hence, assuming that τ_a^* leaves invariant the image of i_2^* , the matrix in Equation (5) is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & \pm i \end{pmatrix},$$

hence it has determinant ± 1 . □

3 Invariance when $\mathrm{trace}(\mu) = -2$.

We discuss first the case where $\mathrm{trace}(\mu) = -2$. The proof has three parts. In Subsection 3.1 we consider a perfect pairing on $H^1(S; \rho_0)$ (pulling back the

cup product on $H^1(\partial M_k; \rho_k)$ by the isomorphism $H^1(S; \rho_0) \cong H^1(\partial M_k; \rho_k)$. We show that, for $k = 1, 2$, the images of $i_k^* : H^1(M_k; \rho_k) \rightarrow H^1(S; \rho_0)$ are isotropic subspaces. Then in Subsection 3.2 we analyze properties of isotropic planes of $H^1(S; \rho_0) \cong \mathbf{C}^4$, that are viewed as lines in a ruled quadric in \mathbf{P}^3 . The properties of this ruled quadric are used in a deformation argument in Subsection 3.3 to conclude the proof when $\text{trace}(\mu) = -2$.

3.1 A perfect pairing

For a closed oriented n -manifold N^n and a representation $\rho : \pi_1(N^n) \rightarrow \text{SL}(2, \mathbf{C})$, there is a nondegenerate pairing:

$$\cup : H^k(N^n; \rho) \times H^{n-k}(N^n; \rho) \rightarrow H^n(N^n; \mathbf{C}^2 \otimes \mathbf{C}^2) \rightarrow H^n(N^n; \mathbf{C}) \cong \mathbf{C}, \quad (6)$$

which is the composition of the cup product and the map induced from the ρ -invariant pairing

$$\begin{aligned} \mathbf{C}^2 \otimes \mathbf{C}^2 &\rightarrow \mathbf{C} \\ \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} &\mapsto \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \end{aligned}$$

This pairing is bilinear, nondegenerate (Poincaré duality) and natural. In addition, when $n = 2$ and $k = 1$ it is symmetric (because both the usual cup product and the determinant are antisymmetric). In particular, we have two symmetric pairings, for $k = 1, 2$:

$$\cup_k : H^1(\partial M_k; \rho_k) \times H^1(\partial M_k; \rho_k) \rightarrow H^2(\partial M_k; \mathbf{C}) \cong \mathbf{C}. \quad (7)$$

The following lemma assumes that $\text{trace}(\rho(\mu)) = -2$, as the whole section, though it only requires $\text{trace}(\rho(\mu)) \neq 2$.

Lemma 3.1. $H^*(\partial S; \rho_0) = 0$.

Proof. Since $\text{trace}(\rho(\mu)) = -2$, $\rho_0(\mu)$ has no nontrivial fixed vectors in \mathbf{C}^2 . Thus $H^0(\partial S; \rho_0) = 0$ and, by duality, $H^*(\partial S; \rho_0) = 0$. \square

Lemma 3.2. For $k = 1, 2$, the inclusion map induces an isomorphism

$$H^1(S; \rho_0) \cong H^1(\partial M_k; \rho_k). \quad (8)$$

Proof. This follows from the Mayer-Vietoris sequence applied to S and $\overline{\partial M_k \setminus S}$, which is the union of the two annuli around the arcs of $K \cap B_k$ that have the homotopy type of a component of ∂S . Hence by Lemma 3.1, $H^*(\partial M_k \setminus S; \rho_k) = 0$. \square

Pulling back the pairings (7) by the isomorphism (8), we obtain two symmetric perfect pairings

$$\cup'_k : H^1(S; \rho_0) \times H^1(S; \rho_0) \rightarrow \mathbf{C}. \quad (9)$$

Lemma 3.3. *Both pairings are the same: $\cup'_1 = \cup'_2$.*

Proof. We use de Rham cohomology for the proof. We first claim that every cohomology class in $H^1(S; \rho_0)$ is represented by a form compactly supported in the interior of S , using that $H^1(\partial S; \rho_0) = 0$. Namely, let $\omega \in \Omega^1(S; \rho_0)$ be any closed form and $U \subset S$ a tubular neighborhood of ∂S . Since $H^1(\partial S; \rho_0) = 0$, the restriction of ω to U is exact: there exists a section $s \in \Omega^0(U; \rho_0)$ satisfying

$$\omega|_U = ds,$$

where d is the differential. Consider a smooth function $\varphi : U \rightarrow [0, 1]$ that extends smoothly to zero on $S \setminus U$ and equals one in a smaller neighborhood of ∂S . Then the form

$$\omega - d(\varphi s)$$

is cohomologous to ω and is supported on a compact subset of the interior of S .

Given two closed forms $\omega_1, \omega_2 \in \Omega^1(S; \rho_0)$ with compact support in the interior of S , we may view them as smooth forms $\tilde{\omega}_1, \tilde{\omega}_2$ on ∂M_k by extending them trivially. As the isomorphism $H^1(\partial M_k; \rho_k) \cong H^1(S; \rho_0)$ is induced by restriction, it maps the cohomology classes $[\tilde{\omega}_1], [\tilde{\omega}_2] \in H^1(\partial M_k; \rho_k)$ to $[\omega_1], [\omega_2] \in H^1(S; \rho_0)$. Thus the cup product \cup'_k of their cohomology classes is

$$[\omega_1] \cup'_k [\omega_2] = [\tilde{\omega}_1] \cup_k [\tilde{\omega}_2] = \int_{\partial M_k} \det(\tilde{\omega}_1 \wedge \tilde{\omega}_2),$$

where $\det(\tilde{\omega}_1 \wedge \tilde{\omega}_2)$ denotes the pairing associated to the determinant evaluated at the exterior product of the forms. By construction, the support of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ is contained in S , thus the previous integral can be evaluated on S instead of ∂M_k , and in particular it does not depend on k . \square

By the previous lemma, we may omit the subindex and just write

$$\cup = \cup'_1 = \cup'_2.$$

Lemma 3.4. *The image of $i_2^* : H^1(M_2; \rho_2) \rightarrow H^1(S; \rho_0)$ is an isotropic plane for the product (9).*

Proof. The fact that the image is a plane follows from Lemma 2.2 and its proof. Then, by construction the lemma is equivalent to saying that the image of $H^1(M_k; \rho_2) \rightarrow H^1(\partial M_k; \rho_k)$ is an isotropic subspace. This is well known [8, 16], but we sketch the argument for completeness.

Using the long exact sequence of the pair $(M_k, \partial M_k)$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^1(M_k; \rho_k) & \xrightarrow{j_k^*} & H^1(\partial M_k; \rho_k) & \xrightarrow{\Delta} & H^2(M_k, \partial M_k; \rho_k) \\
\times & & \times & & \times \\
H^2(M_k, \partial M_k; \rho_k) & \xleftarrow{\Delta} & H^1(\partial M_k; \rho_k) & \xleftarrow{j_k^*} & H^1(M_k; \rho_k) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{C} & & \mathbf{C} & & \mathbf{C}
\end{array} \tag{10}$$

where the columns denote the pairings. This implies

$$j_k^*(a) \cup b = a \cup \Delta(b), \quad \forall a \in H^1(M_k; \rho_k) \text{ and } b \in H^1(\partial M_k; \rho_k).$$

Hence

$$j_k^*(a) \cup j_k^*(b) = a \cup \Delta(j_k^*(b)) = 0, \quad \forall a, b \in H^1(M_k; \rho_k),$$

because $\Delta \circ j_k^* = 0$, and we are done. \square

3.2 Finding isotropic planes with the ruled quadric

Let \mathbf{P}^3 denote the projective space on $H^1(S; \rho_0) \cong \mathbf{C}^4$. Isotropic planes of $H^1(S; \rho_0)$ are in bijection with projective lines in the quadric

$$Q = \{x \in \mathbf{P}^3 \mid x \cup x = 0\}.$$

Since \cup is a nondegenerate pairing, Q is the standard quadric, which is a ruled surface, with two rulings. We recall next its basic properties.

Proposition 3.5. *There are two disjoint families of projective lines \mathcal{L}_+ and \mathcal{L}_- in Q such that:*

- (i) *Every line in Q belongs to either \mathcal{L}_+ or \mathcal{L}_- .*
- (ii) *Every point in Q belongs to precisely one line in \mathcal{L}_+ and one in \mathcal{L}_- .*
- (iii) *Two lines in Q intersect if, and only if, one is in \mathcal{L}_+ and the other one is in \mathcal{L}_- .*
- (iv) *Embedding those spaces of lines in the projective Grassmannian, $\mathcal{L}_+ \cong \mathcal{L}_- \cong \mathbf{P}^1$.*

We shall also use the action of $\mathrm{SO}(4, \mathbf{C})$, the isometry group of $H^1(S; \rho_0)$ equipped with the quadratic form \cup in (9). Recall the isomorphism:

$$\mathrm{SL}(2, \mathbf{C}) \times \mathrm{SL}(2, \mathbf{C}) / \pm (\mathrm{Id}, \mathrm{Id}) \cong \mathrm{SO}(4, \mathbf{C}).$$

After projectivizing, this induces an isomorphism:

$$\mathrm{PSL}(2, \mathbf{C}) \times \mathrm{PSL}(2, \mathbf{C}) \cong \mathrm{PSO}(4, \mathbf{C}).$$

Proposition 3.6. *The action of $\mathrm{PSL}(2, \mathbf{C}) \times \mathrm{PSL}(2, \mathbf{C})$ is equivalent to the product action on $\mathcal{L}_+ \times \mathcal{L}_- \cong \mathbf{P}^1 \times \mathbf{P}^1$ (by an equivalence that preserves the product).*

See [6] for a proof. It follows from this proposition that $\mathrm{PSL}(2, \mathbf{C}) \times \{\mathrm{Id}\}$ acts trivially on \mathcal{L}_- , and $\{\mathrm{Id}\} \times \mathrm{PSL}(2, \mathbf{C})$ acts trivially on \mathcal{L}_+ .

Consider the involutions τ_1, τ_2 and τ_3 of the Conway sphere C , as in Figure 1. For $i = 1, 2, 3$ let $a_i \in \mathrm{SL}(2, \mathbf{C})$, satisfy $\rho_0 \circ \tau_{i*} = \rho_0^{a_i}$, and define $\tau_{a_i}^* = a_{i*} \circ \tau_i^*$ the corresponding actions on $H^1(S; \rho_0)$.

Lemma 3.7. *The induced maps $\tau_{a_i}^*$ on \mathbf{P}^3 lie in one factor $\mathrm{PSL}(2, \mathbf{C}) \times \{\mathrm{Id}\}$ or $\{\mathrm{Id}\} \times \mathrm{PSL}(2, \mathbf{C})$. In addition all $\tau_{a_1}^*, \tau_{a_2}^*$ and $\tau_{a_3}^*$ lie in the same factor.*

Proof. By looking at the action on $H^1(S; \rho_0) \cong \mathbf{C}^4$, we have $(\tau_{a_i}^*)^2 = (a_{i*})^2 \circ (\tau_i^*)^2 = -\mathrm{Id}$, by (4). This implies that $\tau_{a_i}^*$ projects to an involution of \mathbf{P}^3 preserving \cup , hence to an involution in $\mathrm{PSL}(2, \mathbf{C}) \times \mathrm{PSL}(2, \mathbf{C})$. Notice that if an involution of $\mathrm{PSL}(2, \mathbf{C}) \times \mathrm{PSL}(2, \mathbf{C})$ is nontrivial on each factor, it lifts to an element of $\mathrm{SL}(2, \mathbf{C}) \times \mathrm{SL}(2, \mathbf{C})$ whose square is $-(\mathrm{Id}, \mathrm{Id})$, hence to an involution of $\mathrm{SO}(4, \mathbf{C}) \cong \mathrm{SL}(2, \mathbf{C}) \times \mathrm{SL}(2, \mathbf{C}) / \pm(\mathrm{Id}, \mathrm{Id})$. As $(\tau_{a_i}^*)^2 = -\mathrm{Id} \in \mathrm{SO}(4, \mathbf{C})$, we deduce that each $\tau_{a_i}^*$ projects to an involution of one of the factors of $\mathrm{PSL}(2, \mathbf{C}) \times \mathrm{PSL}(2, \mathbf{C})$, and is trivial on the other factor. This proves the first assertion of the lemma. For the last assertion, just use that $\tau_{a_1}^* \tau_{a_2}^* = \pm \tau_{a_3}^*$. \square

Hence, up to permuting \mathcal{L}_- and \mathcal{L}_+ , we get:

Corollary 3.8. (i) $\tau_{a_1}^*, \tau_{a_2}^*$ and $\tau_{a_3}^*$ act trivially on \mathcal{L}_- .

(ii) There is no point in \mathcal{L}_+ fixed by all $\tau_{a_1}^*, \tau_{a_2}^*$ and $\tau_{a_3}^*$.

Notice that, for assertion (ii), we use that the subgroup of $\mathrm{PSL}(2, \mathbf{C})$ consisting of three involutions and the identity (also called the 4-Klein group) has no global fixed point in \mathbf{P}^1 .

Let Im denote the image. Since $\mathrm{Im}(i_1^*) \oplus \mathrm{Im}(i_2^*) = H^1(S; \rho_0)$, from Proposition 3.5 (iii) we deduce:

Corollary 3.9. *Either both $\mathrm{Im}(i_1^*)$ and $\mathrm{Im}(i_2^*)$ belong to \mathcal{L}_- , or they both belong to \mathcal{L}_+ .*

Either $\mathrm{Im}(i_2^*)$ belongs to \mathcal{L}_- and, by Corollary 3.8, we may apply Proposition 2.3, or $\mathrm{Im}(i_2^*)$ belongs to \mathcal{L}_+ . To get rid of this last case we will use a deformation argument. The idea is to deform the hyperbolic structure on M_2 , so that it matches with another tangle which is invariant under the involutions τ_i . By Corollary 3.8(ii), the tangle invariant by the involutions satisfies $\mathrm{Im}(i_1^*) \in \mathcal{L}_-$, hence $\mathrm{Im}(i_2^*) \in \mathcal{L}_-$ for the deformed structure on M_2 , by Corollary 3.9. Then we shall use a continuity argument to have the same conclusion for the initial structure on M_2 . Next subsection is devoted to this deformation argument.

3.3 A deformation argument

Let $A = \overline{\partial M_2 \setminus S}$ be the pair of annuli, one around each arc of $K \cap B_2$. The pair (M_2, A) is a *pared* manifold.

Definition. A *pared* manifold is a pair (N, P) , where N is a compact oriented 3-manifold, $P \subset \partial N$ is a union of tori and annuli, such that

- (i) no two components of P are isotopic in ∂N ,
- (ii) every abelian noncyclic subgroup of $\pi_1(N)$ is conjugate to a subgroup of a component of P , and
- (iii) there are no essential annuli $(S^1 \times [0, 1], S^1 \times \partial[0, 1]) \rightarrow (M, P)$.

We say that a pared manifold (N, P) is hyperbolic when the interior of N admits a complete hyperbolic structure with cusps at P . The rank of the cusp is one for an annulus, and two for a torus.

Lemma 3.10. *There exists a pared manifold (M_3, A') , such that*

- 1. *(M_3, A') is obtained from a 2-tangle: namely M_3 is the exterior of two properly embedded arcs in a 3-ball, A' are the annuli around the arcs of the tangle, and $A' \cup S = \partial M_3$.*
- 2. *For $i = 1, 2, 3$, $\tau_i : S \rightarrow S$ extends to an involution of (M_3, A') .*
- 3. *The pared manifolds (M_3, A') and $(M_2 \cup M_3, A \cup A')$ are both hyperbolic.*

Proof of Lemma 3.10. We take M_3 to be the exterior of a simple 2-tangle that is symmetric with respect to τ_1 , τ_2 and τ_3 . Here simple means that M_3 is irreducible, ∂ -irreducible, atoroidal and anannular. In [22], Wu gives a criterion for deciding when a rational tangle is simple and provides examples of simple tangles with the required symmetries. In particular, the pared manifold (M_3, A') admits a hyperbolic structure with totally geodesic boundary in $S = \partial M_3 \setminus A'$ (and rank one cusps in A'). Since (M_3, A') is simple and (M_2, A) hyperbolic, standard arguments in 3-dimensional topology prove that $(M', T') = (M_2 \cup M_3, A \cup A')$ is irreducible, acylindrical, atoroidal and not Seifert fibered (S should be horizontal in a Seifert fibration), hence hyperbolic. \square

The variety of representations of $\pi_1(M_2)$ to $\mathrm{SL}(2, \mathbf{C})$ is denoted by

$$R(M_2) = \mathrm{hom}(\pi_1(M_2), \mathrm{SL}(2, \mathbf{C})),$$

and it is an algebraic subset of affine space \mathbf{C}^N .

Lemma 3.11. *If $\mathrm{trace}(\rho(\mu)) = -2$, then $\mathrm{Im}(i_2^*)$ belongs to \mathcal{L}_- .*

Proof. We connect $\rho_2 \in R(M_2)$ to $\rho'_2 \in R(M_2)$, a lift of the holonomy representation of M_2 that matches with the tangle M_3 of Lemma 3.10, which is a symmetric tangle. Namely we want to find a path or representations

$$\begin{aligned} [0, 1] &\rightarrow R(M_2) \\ t &\mapsto \varphi_t \end{aligned}$$

that satisfies:

- (i) $\varphi_0 = \rho_2$.
- (ii) $\forall t \in [0, 1]$, φ_t is the lift of the holonomy of a hyperbolic structure on M_2 , with rank one cusps at the arcs $K \cap B_2$, and satisfying

$$\text{trace}(\varphi_t(\mu)) = -2.$$

- (iii) $\forall t \in [0, 1]$, $\dim H_1(M_2; \varphi_t) = 2$.
- (iv) $\varphi_1 = \rho'_2$ is the lift of the holonomy of a hyperbolic structure on M_2 that matches with M_3 in Lemma 3.10.

Assuming we have this path of representations, then since M_3 is τ_1 and τ_2 -invariant, the image of $i_3^* : H^1(M_3; \varphi_1) \rightarrow H^1(S; \varphi_1|_{\pi_1(S)})$ is a subspace $\tau_{a_i}^*$ -invariant. Hence the image of i_3^* must be contained in \mathcal{L}_- , by Corollary 3.8 (ii). This implies that for this hyperbolic structure

$$\text{Im}(i_2^* : H^1(M_2; \rho'_2) \rightarrow H^1(S; \rho'_2|_S)) \in \mathcal{L}_-,$$

by Corollary 3.9. Now, since there exists the path φ_t , the ruled quadric of $H^1(S_0; \varphi_t)$ is also deformed continuously (notice that as $\varphi_t|_{S_0}$ is irreducible and φ_t of a meridian has trace -2 , by (iii), Lemmas 2.1, 3.1, and 3.4 apply to $H^1(S_0; \varphi_t)$). Hence along the deformation, the image of i_2^* is contained in \mathcal{L}_- , as $\mathcal{L}_+ \cap \mathcal{L}_- = \emptyset$. Hence

$$\text{Im}(i_2^* : H^1(M_2; \rho_2) \rightarrow H^1(S; \rho_0)) \in \mathcal{L}_-,$$

as claimed.

Let us justify the existence of the path ϕ_t between ρ_2 and ρ'_2 . If both $\rho_2(\pi_1(M_2))$ and $\rho'_2(\pi_1(M_2))$ are geometrically finite, then they can be connected along the space of geometrically finite structures of the pared manifold, because by Ahlfors-Bers theorem this space is isomorphic to the Teichmüller space of S , cf. [13]. In addition, this is an open subset of the variety of representations, and since the dimension of de cohomology is upper semi-continuous (it can only jump in a Zariski closed subset), (iii) can be achieved by avoiding a proper Zariski closed subset (hence of real codimension ≥ 2). If any of $\rho_2(\pi_1(M_2))$ and $\rho'_2(\pi_1(M_2))$ is not geometrically finite, then it lies in the closure of geometrically finite structures (cf. [12] though this is a particular case of the density theorem), thus there is still a path in the space of representations satisfying (ii) and (iii). \square

By Lemma 3.11, Corollary 3.9 and Proposition 2.3,

$$\mathrm{tor}(M, \rho) = \pm \mathrm{tor}(M^\tau, \rho^\tau).$$

We shall prove that there is also equality of signs:

Proposition 3.12. *If $\mathrm{trace}(\rho(\mu)) = -2$, then*

$$\mathrm{tor}(M, \rho) = \mathrm{tor}(M^\tau, \rho^\tau).$$

Proof. To remove the sign ambiguity, we use again the deformation φ_t of the proof of Lemma 3.11. Since $\forall t \in [0, 1]$, φ_t satisfies the sufficiency criterion of Proposition 2.3, the eigenvalues of $\tau_{a_i}^*$ restricted to the image of i_2^* belong to $\{\pm i\}$, and they do not change as we deform t , hence the determinant of $\tau_{a_i}^*$ restricted to the image of i_2^* is $+1$, because this holds for $\rho'_2 = \varphi_1$ (as M_3 is τ_i -invariant). \square

4 Invariance when $\mathrm{trace}(\rho(\mu)) = +2$.

When $\mathrm{trace}(\rho(\mu)) = 2$, then Lemma 3.1 does not apply, hence we cannot use the argument of Section 3. Recall that

$$R(M) = \mathrm{hom}(\pi_1(M), \mathrm{SL}(2, \mathbf{C}))$$

denotes the variety of representations of $\pi_1(M)$ in $\mathrm{SL}(2, \mathbf{C})$. We will consider representations $\rho_n \in R(M)$ to which the arguments of Section 3 apply and such that ρ_n converges to ρ in $R(M)$, as $n \rightarrow \infty$.

Let $\rho \in R(M)$ be a lift of the holonomy with $\mathrm{trace}(\rho(\mu)) = 2$. By Thurston's hyperbolic Dehn filling and for $n \in \mathbf{N}$ large enough, the orbifold with underlying space S^3 , branching locus K and ramification index n is hyperbolic. It induces a representation of $\pi_1(M)$ in $\mathrm{PSL}(2, \mathbf{C})$ that lifts to $\rho_n \in R(M)$. The lift satisfies $\mathrm{trace}(\rho_n(\mu)) = \pm 2 \cos(\pi/n)$, and there is precisely one lift for every choice of sign. We chose the lift satisfying $\mathrm{trace}(\rho_n(\mu)) = +2 \cos(\pi/n)$.

Proposition 4.1 ([17]). *For $n \in \mathbf{N}$ large enough, there exist $\rho_n \in R(M)$ which is a lift of the holonomy of the orbifold with underlying space S^3 , branching locus K and ramification index n , so that $\rho_n \rightarrow \rho$.*

These orbifolds can also be considered for the mutant knot, and there exist the corresponding mutant representations

$$\rho_n^\tau \in R(M^\tau).$$

Namely, the lifts of the holonomies of the orbifold structures on K^τ are the “mutant representations” of ρ_n . Moreover, $\rho_n^\tau \rightarrow \rho^\tau$.

Lemma 4.2. *For $n \in \mathbf{N}$ large enough, $H^*(M; \rho_n) \cong H^*(M^\tau; \rho_n^\tau) = 0$ and*

$$\mathrm{tor}(M, \rho_n) = \mathrm{tor}(M^\tau, \rho_n^\tau).$$

Proof. We use semi-continuity of cohomology to say that $H^*(M; \rho_n) \cong H^*(M^\tau; \rho_n^\tau) = 0$. More precisely, the dimension of cohomology is an upper semi-continuous function on $R(M)$, cf. [7]. Hence, as ρ and ρ^τ are acyclic, then all representations in a Zariski open subset containing ρ and ρ^τ are acyclic, and so are ρ_n and ρ_n^τ , as claimed.

Since ρ_n and ρ_n^τ are acyclic, $\dim H_1(M_2; \rho_n) = \dim H_1(M_2; \rho_n^\tau) = 2$. In addition, up to conjugation,

$$\rho_n(\mu) \sim \begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix}.$$

Hence \mathbf{C}^2 has no $\rho_n(\mu)$ -invariant proper subspaces and Lemma 3.1 applies. Thus the pairing (9), Lemma 3.4 and all the results of Section 3.1 hold true for ρ_n . To conclude, for the deformation argument, we use that ρ_n is the lift of the holonomy of an orbifold. Instead of working with pared structures on (M_2, A) , we work with orbifold structures with underlying space the ball B_2 , branching locus $K \cap B_2$ and branching index n . The results on the space of hyperbolic structures (geometrically finite or infinite) apply, and we may use the deformation argument of Lemma 3.11. \square

By Proposition 4.1 and Lemma 4.2, by taking the limit when $n \rightarrow \infty$ we get:

Corollary 4.3. *If $\mathrm{trace}(\rho(\mu)) = +2$, then*

$$\mathrm{tor}(M, \rho) = \mathrm{tor}(M^\tau, \rho^\tau).$$

The proofs for $\mathrm{trace}(\rho(\mu)) = 2$ and $\mathrm{trace}(\rho(\mu)) = -2$ are quite different, because $H^1(\partial S_0; \rho_0)$ is non zero when the trace is $+2$, and vanishes when it is -2 . The generic case is $\mathrm{trace}(\mu) \neq 2$. The proof of Section 3 applies to the following situation.

Proposition 4.4. *Let $\rho : \pi_1(M) \rightarrow \mathrm{SL}(2, \mathbf{C})$ be a representation satisfying:*

1. $H^1(M; \rho) = 0$;
2. $\mathrm{trace}(\rho(\mu)) \neq 2$;
3. ρ restricted to $\pi_1(S)$ is irreducible;
4. the representation ρ is in the same irreducible component of $R(M)$ as some representation such that the image of i_2^* is contained in Q_- .

Then $\mathrm{tor}(M, \rho) = \mathrm{tor}(M^\tau, \rho)$.

Corollary 4.5. *For a generic representation ρ of the irreducible component of $R(M)$ that contains a lift of the holonomy, $\text{tor}(M, \rho) = \text{tor}(M^\tau, \rho^\tau)$.*

Question. *The holonomy representation of a hyperbolic knot has two lifts to $\text{SL}(2, \mathbb{C})$, each one with a different sign for the image of the meridian. Do they belong to the same irreducible component of the variety of representations?*

This happens to be true for instance if the component of the variety of representations contains a dihedral representation, as this is a ramification point or the map from the variety of representations in $\text{SL}(2, \mathbb{C})$ to those in $\text{PSL}(2, \mathbb{C})$.

5 Example: Kinoshita-Terasaka and Conway mutants

Let KT and C be the Kinoshita-Terasaka knot and the Conway knot respectively. It is well known that they are mutant hyperbolic knots. Using the Snap program [3], based of J. Weeks' SnapPea [21], we have obtained all the necessary information to compute their torsion.

The fundamental groups of these knots have the following presentations:

$$\begin{aligned}\pi_1(S^3 \setminus C) &= \langle abc \mid abACbcbacBCABaBc, aBcBCABacbCbAbacbc \rangle, \\ \pi_1(S^3 \setminus KT) &= \langle abc \mid aBCbABBCbaBcbbcABcbbaB, abcACaB \rangle.\end{aligned}$$

As usual, capital letters denote inverse.

The image of the holonomy representation is contained in $\text{PSL}(2, \mathbf{Q}(\omega))$ where $\mathbf{Q}(\omega)$ is the number field generated by a root ω of the following polynomial:

$$p(x) = x^{11} - x^{10} + 3x^9 - 4x^8 + 5x^7 - 8x^6 + 8x^5 - 5x^4 + 6x^3 - 5x^2 + 2x - 1.$$

The torsions then are elements of $\mathbf{Q}(\omega)$. In order to express elements in $\mathbf{Q}(\omega)$, we use the \mathbf{Q} -basis $(\omega^{10}, \omega^9, \dots, \omega, 1)$. Tables 1 and 2 give the coefficients of the torsions of KT and C with respect to this \mathbf{Q} -basis. On each table, the first column gives the element of the basis. We let n denote the dimension of the irreducible representation of $\text{SL}(2, \mathbb{C})$ used to compute the torsion, and the tables show the values for $n = 2$ (i.e. the standard representation), but also $n = 4$ and $n = 6$. In order to compare them, the coefficients of the torsion for Kinoshita-Terasaka (KT) and Conway (C) knots are tabulated side by side. We give a table for each lift of the holonomy, one when the trace of the meridian is 2 (Table 1) and another when it is -2 (Table 2).

Of course, for $n = 2$ and for any lift of the holonomy, the torsion of KT and the torsion of C is the same. Notice that for the 4-dimensional representation, they are also the same for one lift but different for the other, and that they differ for both lifts when we use the 6-dimensional representation.

	$n = 2$		$n = 4$		$n = 6$	
	KT	C	KT	C	KT	C
ω^{10}	356	356	11112880	11112880	676803770859632	662357458754672
ω^9	-620	-620	-38963592	-38963592	-640579476284656	-579216259622896
ω^8	636	636	36107416	36107416	212555254795952	153724448856752
ω^7	-864	-864	-31579196	-31579196	-990061444305088	-943617945204928
ω^6	1228	1228	60889040	60889040	1004678681648016	908722528184976
ω^5	-1080	-1080	-58195768	-58195768	-444238765345264	-349679698188784
ω^4	780	780	36555000	36555000	482101712163904	424247992815424
ω^3	-628	-628	-31740272	-31740272	-371824600930944	-320894530449024
ω^2	428	428	21313180	21313180	51168266257072	15655188602032
ω^1	-188	-188	-8829332	-8829332	-165869512283168	-152117462516768
1	124	124	7476160	7476160	-37602419304496	-50452054740016

Table 1: Torsions for the lift of the holonomy with trace of the meridian 2. The table gives the coefficients of the torsion of n -dimensional representation (with respect to a \mathbf{Q} -basis for $\mathbf{Q}(\omega)$).

	$n = 2$		$n = 4$		$n = 6$	
	KT	C	KT	C	KT	C
ω^{10}	7352	7352	-106244812	-84923788	-5089618734386048	-5181970358958464
ω^9	12100	12100	-40892392	-98464552	26333637242897408	26767528167113984
ω^8	-18868	-18868	135740632	176373400	-26132678464882128	-26556943437149136
ω^7	-16	-16	81031412	30483572	18961525460403712	19282331500463872
ω^6	-19124	-19124	70025564	154082012	-41268295304316624	-41948393922548432
ω^5	29448	29448	-188927128	-264857368	41815776250571680	42495766908786848
ω^4	-14272	-14272	71097428	118825172	-25207995553964480	-25621419777084608
ω^3	13576	13576	-71628932	-116091140	22311420427155024	22676270315709264
ω^2	-13352	-13352	98553148	124139068	-15990083236426320	-16248280122238544
ω^1	2780	2780	-4562444	-18136844	5898804809613840	5996288593045520
1	-5812	-5812	48068144	56560304	-5891958922292320	-5986195442605152

Table 2: Torsions for the lift of the holonomy with trace of the meridian -2 , for the n -dimensional representations. Again the table gives the coefficients with respect to a \mathbf{Q} -basis for $\mathbf{Q}(\omega)$.

As said in the introduction, when $n = 2$, these had been computed by Dunfield, Friedl and Jackson in [4]. They computed numerically a twisted Alexander invariant (which are not mutation invariant) for all knots up to 15 crossings, and the torsions computed here are just the evaluations at ± 1 .

6 Other mutations and other representations

The mutation considered in this paper is called $(0, 4)$ -mutation, because the involved surface is planar and has 4 boundary components. By tubing along invariant arcs of the knot, this is a particular case of the so called $(2, 0)$ -mutation, namely, the mutation along a closed surface of genus 2 and using the hyperelliptic involution.

In [15] Ruberman proved that a $(2, 0)$ -mutant of a hyperbolic manifold is again hyperbolic. The behavior of invariants under $(2, 0)$ -mutation has been investigated by many authors, see [1, 5, 11] for instance. Unfortunately, our arguments do not apply, as in Section 3 we require two involutions, and in genus two mutation we can use only the hyperelliptic involution. So we arise:

Question. *Is $\text{tor}(M, \rho)$ invariant under genus two mutation?*

The three dimensional representation of $\text{SL}(2, \mathbf{C})$ is conjugate to the adjoint representation in the automorphism group of the Lie algebra $\mathfrak{sl}(2, \mathbf{C})$. The representation $\text{Ad}\rho$ is not acyclic, but a natural choice of basis for homology has been given in [14], hence its torsion is well defined. Moreover, we have:

Proposition 6.1 ([14]). *The torsion $\text{tor}(M, \text{Ad}\rho)$ is invariant under $(2, 0)$ -mutation.*

The proof is straightforward, as $H^1(S; \text{Ad}\rho_0)$ is the cotangent space to the variety of characters of S , and the action of the hyperelliptic involution is trivial on the variety of characters of S .

We have seen that if we compose the lift of the holonomy with the 6-dimensional representation of $\text{SL}(2, \mathbf{C})$ (or the 4-dimensional one when the trace of the meridian is -2), the torsion is not invariant under $(2, 0)$ -mutation, as it is not invariant under $(0, 4)$ -mutation, see the example of the previous section.

Question. *Working with the lift of the holonomy with trace of the meridian 2, is the torsion of the 4-dimensional representation invariant under $(0, 4)$ -mutation?*

To conclude, we notice that our arguments do not apply if we tensorize $\rho : \pi_1(M) \rightarrow \text{SL}(2, \mathbf{C})$ with the abelianization map $\pi_1(M) \twoheadrightarrow \mathbf{Z} = \langle t \rangle$. This torsion gives the twisted polynomial in $\mathbf{C}[t^{\pm}]$ studied in [4], where it

is proved not to be mutation invariant. If we could apply the arguments of this paper, then we would be in the generic situation of Proposition 4.4 and Section 3, because $\text{trace}(\rho(\mu)) = \pm(t + 1x/t) \neq 2$. Notice that two of the involutions in Figure 1 reverse the orientation of the meridian, and only one preserves them. Thus we can use only a single involution, and at least two involutions are required in our argument from Section 3, more precisely in Corollary 3.8(ii).

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